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Approximation and Strong Approximation in Normed Spaces via Tangent Functionals*

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1. INTRODUCTION

Throughout this paper B denotes a normed space over the real field R, M is a closed subspace of B, and C a convex set in B. The main object of approximation theory amounts to the solution of this problem: Given M (or C), and an element $x \in B$, find elements x_0 in M (respectively in C) such that

$$\| x_0 - x \| \leqslant \| x - y \|, \qquad \text{for every} \quad y \in M \text{ (or every } y \in C). \tag{1}$$

P_M (or P_C) will denote the (generally multivalued) map associating to $x \in B$ the elements defined by (1), when existing. We set also $d = \text{distance}(x, C)$. Recently (see [6]) another kind of approximation from a subspace M has been defined, which naturally extends to any set. This paper studies the

PROBLEM. Given $x \in B$, find elements $x^0 \in M$ (or $x^0 \in C$), called approximations to x, such that

$$\| x^0 - y \| \leqslant \| x - y \| \qquad \text{for every} \quad y \in M \text{ (or every } y \in C), \tag{2}$$

and "strong approximations" to x in this sense which are defined in Section 3. R_M (or R_C) will denote the (generally multivalued) map associating to $x \in B$, its approximations as defined by (2) when existing.

If $x \perp y$ for $x, y \in B$ means $\| x \| \leqslant \| x + ty \|$ for all real t, then the usual problem of best approximation (see [12, I.1.14]) is to find x_0 such that $(x - x_0) \perp M$ and the problem considered here is to find x^0 such that $M \perp (x - x^0)$. In Hilbert space $P_M = R_M$ (see [7]).

In Section 2 we show that the parallels existing between maps R_M and P_M can only partly be extended to maps R_C and P_C and we also relate to R_C another type of approximation defined in [4].

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Section 3 contains a discussion about strong unicity, introduced in [8] for P_M, and similar concepts that we introduce for the other maps.

Section 4 contains an example in $C[0, 1]$.

In the paper we use extensively the tangent functionals $\tau(x, y)$, defined from $B \times B$ into R in this way:

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\| x + ty \| - \| x \|}{t}.$$

Some properties of these functionals (e.g., $(\| x + ty \| - \| x \|)/t$ is a non-decreasing function of $t \in R$) as well as their form in some spaces can be found in [5, 11]. We recall that $\tau(x, y) = \sup_{f \in J_x} f(y)$ where $J_x = \{f \in X^*; \|f\| = 1, f(x) = \| x \|\}$.

2. Approximation in Convex Sets

We begin recalling the Kolmogorov condition (see, e.g., [12, p. 360 and p. 88]).

Theorem 1. *$x_0 \in P_C(x)$ if and only if $\tau(x - x_0, x_0 - y) \geqslant 0$ for every $y \in C$.*

Corollary 1. *$x_0 \in P_M(x)$ if and only if $\tau(x - x_0, m) \geqslant 0$ for every $m \in M$.*

The map R_C satisfies the following properties similar to the results in [6] and the proofs are immediate:

(i) C is contained in the domain of R_C (the subset of those elements of B for which $R_C \neq \varnothing$); moreover, $R_C(x) = \{x\}$ for every $x \in C$;

(ii) $R_C(x)$ is closed if C is closed;

(iii) $R_C(x)$ is convex for every x;

(iv) if x belongs to the domain of R_C, $R_C(x)$ is bounded. In fact, for any $x^0 \in R_C(x)$ we have: $\| x - x^0 \| \leqslant \| x - y \| + \| y - x^0 \| \leqslant 2 \| x - y \|$ for every $y \in C$; so $\| x - x^0 \| \leqslant 2d$;

(v) if $x^0 \in R_C(x)$, then $x^0 \in R_C(tx + (1 - t)x^0)$ for $t \geqslant 1$; in fact:

$$\begin{aligned} \|(1 - t) x^0 + tx - y \| &\geqslant | t | \cdot \| x - y \| - | 1 - t | \cdot \| x^0 - y \| \\ &\geqslant (| t | - | 1 - t |) \| x^0 - y \| \\ &= (t + (1 - t)) \| x^0 - y \| = \| x^0 - y \|. \end{aligned}$$

For other properties of the maps R_M, see also [10].

We shall see that a proposition similar to Theorem 1 does not hold for the maps R_C; in a certain sense, these maps are too general to be used and characterized: For example, $C \subset R_C(x)$ whenever the diameter of C is smaller than d; moreover, the convex sets are not a natural setting for these maps (see Theorem 3 below).

We shall also consider other maps—the so-called "orthogonal retractions" defined in [4]—and those we shall denote by R_C'; if $x' \in C$, we say that $x' \in R_C'(x)$ if

$$\tau(x' - y, x - x') \geqslant 0 \qquad \text{for every} \quad y \in C. \tag{2'}$$

These maps obviously satisfy the properties (i), (ii), (iv), (v); Corollary 2 below will imply that (iii) is also satisfied.

THEOREM 2. *$x' \in R_C'(x)$ implies $x' \in R_C(x)$, and also $x' \in R_C'(tx + (1 - t)x')$ for $t \geqslant 0$.*

Proof. If (2′) holds we have $\| x' - y + t(x - x')\| \geqslant \| x' - y \|$ for every $t \geqslant 0$, and this (set $t = 1$) implies (2). Moreover, if $t \geqslant 0$ we obtain $\tau(x' - y, tx + (1 - t)\, x' - x') = t\tau(x' - y, x - x') \geqslant 0$.

We now consider two properties which are sufficient that $R_C(x) = R_C'(x)$.

PROPOSITION 1. *Suppose that $R_C(x)$ satisfies.*

(A) If $x^0 \in R_C(x)$, then $x^0 \in R_C(tx + (1 - t)\, x^0)$ for $0 \leqslant t \leqslant 1$. Then $R_C(x) = R_C'(x)$.

Proof. In force of the Theorem 2, we have to proof that $R_C(x) \subset R_C'(x)$; but from $\| x^0 - y \| \leqslant \| tx + (1 - t)\, x^0 - y \| = \| x^0 - y + t(x - x^0)\|$ for $0 \leqslant t \leqslant 1$, we obtain (for every $y \in C$):

$$\tau(x^0 - y, x - x^0) = \lim_{t \to 0^+} \frac{\| x^0 - y + t(x - x^0)\| - \| x^0 - y \|}{t} \geqslant 0.$$

Note how (A) implies that $R_C(x)$ is contained in the boundary of C.

In particular, by Theorem 2 and Proposition 1 we obtain the following

COROLLARY 2. *$x^0 \in R_C'(x)$ if and only if $x^0 \in R_C(tx + (1 - t)\, x^0)$ for $0 \leqslant t \leqslant 1$ (so, in view of* (v), *for every $t \geqslant 0$).*

PROPOSITION 2. *Suppose that $R_C(x)$ satisfies:*

(B) If $x^0 \in R_C(x)$ and $y \in C$, then $(1 - t)\, x^0 + ty \in C$ for $t \geqslant 1$. Then $R_C(x) = R_C'(x)$.

Proof. We have to prove that $R_C(x) \subset R_C'(x)$; if $x^0 \in R_C(x)$ and $y \in C$, then by assumption (B) and (1) we obtain

$$\| x - x^0 + t(x^0 - y)\| = \| x - ((1 - t)\, x^0 + ty)\|$$
$$\geqslant \| x^0 - ((1 - t)\, x^0 + ty)\| = t \,\| x^0 - y \|.$$

Dividing by t and setting $1/t = s$, we obtain $\| s(x - x^0) + x^0 - y \| - \| x^0 - y \| \geqslant 0$ $(0 \leqslant s \leqslant 1)$, so $\tau(x^0 - y, x - x^0) \geqslant 0$.

In the propositions proved so far for R_C and R_C', the convexity of C plays no role and only property (iii) depends on it. So we could use (2), (2′) to define similar maps for a set C' that we do not assume to be convex. For that case, Proposition 2 implies the following:

THEOREM 3. *Let C' be a (not necessarily convex) subset of B such that if y_1 and y_2 belong to C', then also $ty_1 + (1 - t)\, y_2 \in C'$ for $t \geqslant 1$. Then $R'_{C'} = R_{C'}$. In particular, $x^0 \in R_M(x)$ iff $\tau(m, x - x^0) \geqslant 0$ for every $m \in M$.*

Remarks. In Hilbert spaces, $R_C = R_C' = P_C$ for every C. If B is two-dimensional and C is closed, then R_C' (so also R_C) is always defined (see [4, Theorem 5]); in particular, R_C' exists whenever C is contained in a one-dimensional subspace of X (this fact is contained in [6, Lemma 1d]). If B is smooth, then R_C' is single-valued and nonexpansive on its domain (see [4, Lemma 1 and Theorem 1]: In that terminology, R_C is a nonexpansive projection); we note that if C is bounded and R_C is defined on B, the fulfillment of (A) for every $x \in B$ is a very strong condition (see [3, 7]).

3. STRONG APPROXIMATION

Now we want to consider problems of "strong approximation," suggested by [2]. We start with the maps of best approximation; following [2] we introduce:

DEFINITION 1. We say that x_0 is "strongly unique," or belongs to $P_C(x)$ (or to $P_M(x)$) strongly, if there exists an $r > 0$ $(r \leqslant 1)$ such that

$$\| x - y \| \geqslant \| x - x_0 \| + r \,\| x_0 - y \| \quad \text{for every } y \in C \text{ (or every } y \in M). \tag{3}$$

Now (3) says that if y moves in C (or in M) from x_0, then the approximation of x worsens with the rate of the distance from x_0.

If $y \in C$, then $z = (1 - t)x_0 + ty \in C$ for $0 \leqslant t \leqslant 1$, so using (3) for z we obtain

$$\|x - x_0 + t(x_0 - y)\| - \|x - x_0\| \geqslant r\|x_0 - z\| = rt\|x_0 - y\| \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad y \in C,$$

that is,

$$\tau(x - x_0, x_0 - y) \geqslant r\|x_0 - y\| \qquad \text{for every } y \in C \text{ (or every } y \in M). \tag{3'}$$

Conversely, from (3') we have: $\|x - x_0 + t(x_0 - y)\| - \|x - x_0\| \geqslant r \cdot \|x_0 - y\|$ for every $t \geqslant 0$, so also for $t = 1$, which is (3).

So (3) and (3') are equivalent, and for a subspace M they become

$$\tau(x - x_0, m) \geqslant r\|m\| \qquad \text{for every} \quad m \in M. \tag{3''}$$

The above definition was introduced in [8] and studied in detail in [2]. Before considering the other maps, we reformulate (using (3')) Lemma 2 of [2].

THEOREM 4. *x_0 belongs to $P_C(x)$ strongly iff the set $A = \{y \in C; \tau(x - x_0, x_0 - y) < \|x - x_0\|\}$ is bounded.*

Proof. If (3') holds, then A is contained in the ball of radius $\|x - x_0\|/r$, centered at x_0: In fact, suppose $\|y - x_0\| > \|x - x_0\|/r$; then we have $\tau(x - x_0, x_0 - y) \geqslant r\|y - x_0\| \geqslant \|x_0 - x\|$, and so $y \notin A$. Conversely, suppose that A is bounded, and that $z \notin A$ for $\|x_0 - z\| \geqslant q > 0$; then, for any $y \neq x_0$ in C, letting $z = x_0 - ((x_0 - y)/\|x_0 - y\|)q$ we have $z \notin A$, and then $\tau(x - x_0, ((x_0 - y)/\|x_0 - y\|)q) \cdot \|x_0 - y\|/q \geqslant \|x - x_0\| \cdot \|x_0 - y\|/q$; so (3') holds with $r = \|x - x_0\|/q$ ((3') trivially holds for $y = x_0$).

Now we want to speak of "strong approximation" for the maps R_C; the concept of strongness we shall introduce for them has a different meaning from that of "strong unicity" for P_C, and seems rather to parallel a notion introduced in [9].

DEFINITION 2. We say that $x^0 \in R_C(x)$ (or $x^0 \in R_M(x)$) strongly, if $x \notin C$ (or $x \notin M$) and there exists an $r > 0$ ($r \leqslant 1$) such that

$$\|x^0 - y\| + r\|x^0 - x\| \leqslant \|x - y\| \qquad \text{for every } y \in C \text{ (or for every } y \in M). \tag{4}$$

DEFINITION 2'. We say that $x' \in R_C'(x)$ strongly, if $x \notin C$, and there exists an $r > 0$ ($r \leqslant 1$) such that

$$\tau(x' - y, x - x') \geqslant r\|x - x'\| \qquad \text{for every } y \in C;\ y \neq x'. \tag{4'}$$

Clearly (4′) implies (4); if (4) is satisfied for x^0 and (B) holds, then $ty + (1-t)x^0 \in C$ for $t \geqslant 1$, so $\tau(x^0 - y, x - x^0) = \lim_{t\to\infty}(\| t(x^0 - y) + (x - x^0)\| - \| t(x^0 - y)\|) = \lim_{t\to\infty}(\| x - (ty + (1-t)x^0)\| - \| x^0 - (ty + (1-t)x^0)\|) \geqslant r \| x - x^0 \|$, in particular for $R_M = R_M'$ (4) is equivalent to (4′), and also to

$$\tau(m, x - x^0) \geqslant r \| x - x^0 \| \qquad \text{for every} \quad m \in M,\ m \neq \theta. \qquad (4'')$$

The definition given by (4) means that if a point is moved in C (or in M) from a strong "approximation" x^0, inside the ball of radius $r \| x - x^0 \|$ and centered at x^0, all the points reached are still "approximations." So the above concept of "strongness" has nothing to do with unicity, and the larger r is, the more x moves from x^0.

The proposition which follows gives an upper bound for the (Chebyshev) radius of the set of strong approximations in the sense of (4) (so also for the set defined by (4′)).

PROPOSITION 3. *The radius of the set of elements which belong strongly to $R_C(x)$ for a given r, is not larger than* $(1 - r)d$.

Proof. Given $\epsilon > 0$, take x_ϵ such that $\| x - x_\epsilon \| < d + \epsilon$; if x^0 satisfies (4), use it with $y = x_\epsilon$: we obtain

$$\| x^0 - x_\epsilon \| \leqslant \| x - x_\epsilon \| - r \| x^0 - x \| < d + \epsilon - rd = (1 - r)d + \epsilon.$$

The conclusion follows since ϵ can be taken arbitrarily small.

In general, we see that the radius of $R_C(x)$ is not larger than d. Moreover, if B is smooth we recall that $R_C'(x)$ can contain at most one point, so in that case no element can belong to $R_C'(x)$ strongly for the meaning of "strongness;" the same for $R_M(x)$ (a similar result holds for $P_M(x)$; see [1, Theorem 5]).

The analog for the maps R_C' of Theorem 4 is the following

THEOREM 5. *$x' \in R_C'(x)$ strongly iff the set $A' = \{ y \in C: \tau(x' - y, x - x') < \| y - x' \|\}$ contains no point of a certain sphere of positive radius, centered at x'.*

Proof. If x' satisfies (4′), then $y \notin A'$ for $\| y - x' \| \leqslant \| x - x' \| \cdot r$; conversely, suppose that $z \notin A'$ for $\| z - x' \| \leqslant q$; take $y \neq x'$ in C, and set $z = x' - ((x' - y)/\|x' - y\|)q$; we have $z \notin A'$, and then $\tau((x' - y)/\|x' - y\|)q, x - x') \geqslant q$, so (4′) holds with $r = q/\| x - x' \|$.

Let $\langle x^0, M\rangle$ denote the linear span of x^0 and M. Then the analog of Proposition 1 in [2] is:

PROPOSITION 4. *If x has a strong approximate (in the sense of (2)) x^0 from M, then so does any element in $\langle x, M\rangle$. More precisely $x^0 \in R_M(x)$ implies $kx^0 + y \in R_M(kx + y)$ strongly with the same r for every $y \in M$ and $k \in R$.*

Proof. If $x^0 \in R_M(x)$ strongly and $k \geqslant 0$, then $\tau(m, kx + y - (kx^0 + y)) = k\tau(m, x - x^0) \geqslant kr \| x - x_0 \| = r \| kx + y - (kx^0 + y)\|$ for every $m \in M$, $m \neq \theta$. If $k < 0$, then $\tau(m, kx + y - (kx^0 + y)) = \tau(-m, -k(x - x^0)) = -k\tau(-m, x - x^0) \geqslant -kr \| x - x^0 \| = r \| kx + y - (kx^0 + y)\|$ for every $m \in M$, $m \neq \theta$.

4. AN EXAMPLE

Consider the space $B = C[0, 1]$; let $x : x(t) = t^2$, and M be the one-dimensional subspace generated by the function $y : y(t) = t$; recall that $\tau(x, y) = \sup_{t\in E}$ [signum $x(t)$] $\cdot\, y(t)$, where $E = \{t \in [0, 1]; x(t) = \| x \|\}$ (see [11, Sect. 6]). We calculate $P_M(x)$; set $ay = x_0 \in P_M(x)$: We must have $\| x - ay \| = \inf_{k\in R} \| x - ky \|$, where

$$\begin{aligned} \| x - ky \| = \sup_{0\leqslant t\leqslant 1} | t^2 - kt | &= 1 - k \qquad \text{if} \quad k \leqslant 2(2^{1/2} - 1) \\ &= k^2/4 \qquad \text{if} \quad k \geqslant 2(2^{1/2} - 1) \end{aligned}$$

so the minimum is attained for $k = 2(2^{1/2} - 1)$, and we have: $x_0 = 2(2^{1/2} - 1)\, y$; $d = \| x - x_0 \| = 3 - 2(2)^{1/2}$. $P_M(x)$ is unique, and also strongly unique: In fact, $E = \{1, 2^{1/2} - 1\}$ so $\tau(x - x_0, x_0 - ky) = \max((x_0 - ky)(1), (ky - x_0)(2^{1/2} - 1)) = \max(2(2)^{1/2} - 2 - k, 4(2)^{1/2} - 6 + (2^{1/2} - 1)k) \geqslant (2^{1/2} - 1) \mid 2(2)^{1/2} - 2 - k \mid = (2^{1/2} - 1) \| x_0 - ky \|$. Now we look for $x^0 = \alpha y \in R_M(x)$. For every $k \in R$ we want to have $\| \alpha y - ky \| \leqslant \| x - ky \|$, where the last term has been calculated above: Setting $k = 0$, we see that we must have $| \alpha | \leqslant 1$; but if $\alpha < 1$, for $k = 2$ we should obtain $\| \alpha y - 2y \| = 2 - \alpha > 1 = \| x - 2y \|$. So $R_M(x)$ is the singleton $\{ y\}$, and $x^0 = y$ satisfies $\| y - ky \| = | 1 - k | \leqslant \| x - ky \|$: y does not belong to $R_M(x)$ strongly by the remarks following (4″), and moreover $\tau(x^0 - ky, x - x^0) = 0$ for every $k \in R$.

Now consider the convex set $C = \{ky; -1 \leqslant k \leqslant \frac{3}{2}\}$; then $P_C(x) = 2(2^{1/2} - 1)\, y$, while $\alpha y \in R_C(x)$ for $\alpha \in [\frac{15}{16}, 1]$ (and $\alpha y \in R_C(x)$ strongly if and only if $\alpha \in (\frac{15}{16}, 1)$). But $R_C{}'(x)$ contains only y: In fact, since $\alpha y - ky$ assumes its norm at 1 and $(\alpha y - ky)(1) = | \alpha - k |$, we have

$$\begin{aligned} \tau(\alpha y - ky, x - \alpha y) &= 1 - \alpha \qquad \text{if} \quad \alpha \geqslant k, \\ &= \alpha - 1 \qquad \text{if} \quad \alpha < k, \end{aligned}$$

which is negative, if $\alpha < 1$, for some $k \in [-1, \frac{3}{2}]$. So only y belongs (but not strongly) to $R_C{}'(x)$, which is strictly contained in $R_C(x)$.

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